

# ON DERIVING A BASIS FOR THE VECTOR SPACE OF BOUNDED QUDIT ERROR OPERATORS OVER $\mathbb{C}^d$

COLIN WILMOTT AND PETER WILD

**ABSTRACT.** We derive a basis for the vector space of bounded operators acting on a  $d$ -dimensional system Hilbert space  $\mathbb{C}^d$ . In the context of quantum computation the basis elements are identified as the generalised Pauli matrices - the error generators. As an application, we show how such matrices are used in the teleportation a single qudit.

## 1. INTRODUCTION

The theory of quantum computation continues to advance our understanding of information as established in the seminal work of Shannon (1948) through an innovative analysis of the nature of noise. This development of a quantum mechanical computing framework has redefined quantum computation and inspires discoveries whose very nature lie at the frontier of reality. Constructing a quantum computer is predicated on realising the inherent processing advantage of quantum computation over its classical analogue and on controlling the sensitive quantum interference effects that explain the source of its computational power. However, computations are taken in open quantum systems which produce unwanted interactions between sensitive quantum information and noise in the environment. It is this interaction that results in decoherence - an outcome that destroys quantum information. Unfortunately, decoherence is an inevitable feature of quantum computation, and therefore, it is of fundamental importance that any coupling between information and the environment be controlled. Therefore, to better understand the fundamentals of noise propagation is to understand the formalism of a model that explains it.

In this paper we construct a basis for the space of bounded operators acting on a  $d$ -dimensional quantum system  $\mathbb{C}^d$ . As an application, we generalise the qubit teleportation scheme.

## 2. PRELIMINARIES

Given an arbitrary finite alphabet  $\Sigma$  of cardinality  $d$ , we process quantum information by specifying a state description of a finite dimension quantum space. In particular, the state description of the Hilbert space  $\mathbb{C}^d$ . While the state of an  $d$ -dimensional Hilbert space can be more generally expressed as a linear combination of basis states  $|\psi_i\rangle$ , we write each orthonormal basis state of the  $d$ -dimensional Hilbert space  $\mathbb{C}^d$  to correspond with an element of  $\mathbb{Z}_d$ . In this context the basis

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*Electronic address:* cmwilmott@maths.ucd.ie.

*Electronic address:* P.Wild@rhul.ac.uk.

$\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$  is referred to as the *computational basis*. Therefore, a state  $|\psi\rangle$  of  $\mathbb{C}^d$  is given by

$$(2.1) \quad |\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle,$$

where  $\alpha_i \in \mathbb{C}$  and  $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$ . A *qudit* describes a state in the Hilbert space  $\mathbb{C}^d$ . The state space of an  $n$ -qudit state is the tensor product of the basis states of the single system  $\mathbb{C}^d$ , written  $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$ , with corresponding orthonormal basis states given by

$$(2.2) \quad |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle = |i_1\rangle |i_2\rangle \dots |i_n\rangle = |i_1 i_2 \dots i_n\rangle,$$

where  $i_j \in \mathbb{Z}_d$ . The general state of a qudit in the Hilbert space  $\mathcal{H}$  is then written

$$(2.3) \quad |\psi\rangle = \sum_{(i_1 i_2 \dots i_n) \in \mathbb{Z}_d^n} \alpha_{(i_1 i_2 \dots i_n)} |i_1 i_2 \dots i_n\rangle,$$

where  $\alpha_{(i_1 i_2 \dots i_n)} \in \mathbb{C}$  and  $\sum |\alpha_{(i_1 i_2 \dots i_n)}|^2 = 1$ .

### 3. AN ERROR MODEL

The challenge of quantum information processing is to elicit a reliable form of communication and to maintain such a form in the presence of quantum noise. Noise is a characteristic of the environment associated with an information state and is a property of an open quantum system that subjects an information state to unwanted interactions with the elements of the environment during teleportation. It is inevitable that the communication of an information state will cause interactions with the environment. However, prolonged contact between the information state and environment is soon to suffer in entanglement that degrades the information state resulting in decoherence. Any strategy to stabilize quantum computations from the effects of noise will ultimately be required to deal with both the problems of decoherence and unitary imperfections of channel communication. To this end, we give the following description of error within the environment system.

Given a qudit information state  $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$  of the Hilbert space  $\mathbb{C}^d$ , let us consider an adjoined environment space  $|E\rangle$  endowed with an orthonormal basis of dimension  $d^2$ . We suppose that both the state space of the qudit and the corresponding environment space are initially independent systems. The joint state of the systems  $|\psi\rangle$  and  $|E\rangle$  is then  $|\psi\rangle \otimes |E\rangle$  and its dynamics may be characterised when we further suppose that the joint system evolves according to some unitary operation. Given a unitary operation  $U$ , we write interaction of each basis qudit with the environment under  $U$  as

$$(3.1) \quad \begin{aligned} U(|i\rangle \otimes |E\rangle) &= \sum_{l=0}^{d-1} \gamma_{-i+l, -i} (|i+l\rangle \otimes |e_{-i+l, -i}\rangle) \\ &= \sum_{l=0}^{d-1} |i+l\rangle \otimes \gamma_{-i+l, -i} |e_{-i+l, -i}\rangle, \end{aligned}$$

for  $i \in \{0, \dots, d-1\}$ . By linearity of  $U$ , the dynamics of the joint system  $|\psi\rangle \otimes |E\rangle$  is then

$$\begin{aligned}
 U(|\psi\rangle \otimes |E\rangle) &= U\left(\left(\sum_{i=0}^{d-1} \alpha_i |i\rangle\right) \otimes |E\rangle\right) \\
 &= U\left(\sum_{i=0}^{d-1} \alpha_i (|i\rangle \otimes |E\rangle)\right) \\
 &= \sum_{i=0}^{d-1} \alpha_i U(|i\rangle \otimes |E\rangle) \\
 (3.2) \quad &= \sum_{i=0}^{d-1} \sum_{l=0}^{d-1} \alpha_i |i+l\rangle \otimes \gamma_{-i+l, -i} |e_{-i+l, -i}\rangle.
 \end{aligned}$$

Since  $\frac{1}{d} \sum_{z=0}^{d-1} \omega^{zk} = 1$  if  $z = 0$  and vanishes otherwise then equation (3.2) may be written as

$$\begin{aligned}
 &\frac{1}{d} \sum_{i=0}^{d-1} \sum_{l=0}^{d-1} \left( \alpha_i |i+l\rangle \otimes \left( \sum_{z=0}^{d-1} \sum_{k=0}^{d-1} \omega^{zk} \gamma_{-i+l+z, -i+z} |e_{-i+l+z, -i+z}\rangle \right) \right) \\
 &= \frac{1}{d} \sum_{i=0}^{d-1} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \alpha_i |i+l\rangle \otimes \left( \sum_{z=0}^{d-1} \omega^{zk} \gamma_{-i+l+z, -i+z} |e_{-i+l+z, -i+z}\rangle \right) \right) \\
 &= \frac{1}{d} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \sum_{i=0}^{d-1} \left( \alpha_i |i+l\rangle \otimes \left( \sum_{z=0}^{d-1} \omega^{zk} \gamma_{-i+l+z, -i+z} |e_{-i+l+z, -i+z}\rangle \right) \right) \right) \\
 &= \frac{1}{d} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \sum_{i=0}^{d-1} \left( \omega^{ik} \alpha_i |i+l\rangle \otimes \left( \sum_{z=0}^{d-1} \omega^{-ik} \omega^{zk} \gamma_{-i+l+z, -i+z} |e_{-i+l+z, -i+z}\rangle \right) \right) \right) \\
 &= \frac{1}{d} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \sum_{i=0}^{d-1} \left( \omega^{ik} \alpha_i |i+l\rangle \otimes \left( \sum_{z'=0}^{d-1} \omega^{z'k} \gamma_{z'+l, z'} |e_{z'+l, z'}\rangle \right) \right) \right) \\
 (3.3) \quad &= \frac{1}{d} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \left( \sum_{i=0}^{d-1} \omega^{ik} \alpha_i |i+l\rangle \right) \otimes \left( \sum_{z'=0}^{d-1} \omega^{z'k} \gamma_{z'+l, z'} |e_{z'+l, z'}\rangle \right) \right).
 \end{aligned}$$

An outer product representation describes the set of operators that act on the joint quantum state under  $U$ . The operator  $X_1 = \sum_{i=0}^{d-1} |i+1\rangle \langle i|$  maps  $\alpha_i |i\rangle$  to  $\alpha_i |i+1\rangle$  for  $i \in \{|0\rangle, \dots, |d-1\rangle\}$ , and thus maps  $\sum_{i=0}^{d-1} \alpha_i |i\rangle$  to  $\sum_{i=0}^{d-1} \alpha_i |i+1\rangle$ . Similarly,  $Z_1 = \sum_{i=0}^{d-1} \omega^i |i\rangle \langle i|$  maps  $\alpha_i |i\rangle$  to  $\omega^i \alpha_i |i\rangle$  and correspondingly maps  $\sum_{i=0}^{d-1} \alpha_i |i\rangle$  to  $\sum_{i=0}^{d-1} \omega^i \alpha_i |i\rangle$ . Both  $X_1$  and  $Z_1$  are called the *Weyl Pair* (Weyl (1931)). Consequently, the action of  $U$  on  $|\psi\rangle \otimes |E\rangle$  is described by the set of operators  $X_l Z_k = \sum_{i=0}^{d-1} \omega^{ik} |i+l\rangle \langle i|$ ,  $(l, k) \in \mathbb{Z}_d \times \mathbb{Z}_d$ ,

$$\begin{aligned}
 &\sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \left( \left( \sum_{i=0}^{d-1} \omega^{ik} \alpha_i |i+l\rangle \right) \otimes \frac{1}{d} \left( \sum_{z'=0}^{d-1} \omega^{z'k} \gamma_{z'+l, z'} |e_{z'+l, z'}\rangle \right) \right) \\
 (3.4) \quad &= \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} X_l Z_k |\psi\rangle \otimes \gamma_{lk} |e_{lk}\rangle.
 \end{aligned}$$

Thus, to correctly specify an error model that describes the action of a unitary operator  $U$  on the joint space  $|\psi\rangle \otimes |E\rangle$ , it is necessary that the environment  $|E\rangle$ , associated with an information state in  $\mathbb{C}^d$ , be a Hilbert space of dimension  $d^2$ . Following the action of  $U$  on the joint system, a measurement on the environment is performed with respect to the basis  $|e_{mn}\rangle$ ,  $(m, n) \in \mathbb{Z}_d \times \mathbb{Z}_d$  to diagnose the introduced error in result (3.4). Therefore, equation (3.4) provides the conceptual foundation of quantum error correction. Measurements taken in the environment basis initiate the correction step  $(X_m Z_n)^{-1} = Z_{(-n \bmod d)} X_{(-m \bmod d)}$ .

We now show that the set  $\{X_l Z_k\}$ ,  $(l, k) \in \mathbb{Z}_d \times \mathbb{Z}_d$ ,  $(X_0 Z_0 = I)$  constitutes a basis for the space of bounded operators on  $\mathbb{C}^d$ . As such, the set  $\{X_l Z_k\}$ ,  $(l, k) \in \mathbb{Z}_d \times \mathbb{Z}_d$ , forms a  $d^2$ -dimensional Lie algebra with a  $d \times d$  matrix representation defined by the matrices with entries  $X_{m,n} = \delta_{m,n-l(\bmod d)}$ ,  $Z_{m,n} = \omega^{mk} \delta_{m,n}$ .

**Theorem 3.1.** *Denote by  $\omega$  the primitive  $d^{\text{th}}$  root of unity. Let us consider  $X_i |k\rangle = |k+i \pmod d\rangle$  and  $Z_j |k\rangle = \omega^{kj} |k\rangle$ . Then  $\mathcal{E} = \{X_i Z_j \mid (i, j) \in \mathbb{Z}_d \times \mathbb{Z}_d\}$  is a basis for the space of bounded operators acting on  $\mathbb{C}^d$ .*

*Proof:* To show that elements of  $\mathcal{E}$  are linearly independent and span  $\mathbb{C}^d$ , it suffices to show that the basis  $\{|a\rangle\langle b|\}$ ,  $a, b \in \mathbb{Z}_d$ , on  $\mathbb{C}^d$  may be expanded as a linear combination of elements in  $\mathcal{E}$  as both sets of operators are of size  $d^2$ . Let us consider  $\mathcal{E}$  in the  $\{|a\rangle\langle b|\}$  basis as

$$(3.5) \quad E_{i,j} = \sum_{k=0}^{d-1} \omega^{jk} |k+i\rangle\langle k|$$

then  $E_{i,j} |l\rangle = X_i Z_j |l\rangle = X_i \omega^{jl} |l\rangle = \omega^{jl} |l+i\rangle$ . Suppose we may express  $|a\rangle\langle b|$  as the linear combination  $|a\rangle\langle b| = \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \xi_{i,j} E_{i,j}$ . Then coefficient  $\xi_{i,j}$  is given by

$$\begin{aligned} \xi_{i,j} &= \frac{1}{d} \text{tr} \left( E_{i,j}^\dagger |a\rangle\langle b| \right) \\ &= \frac{1}{d} \text{tr} \left( \sum_{k=0}^{d-1} \omega^{-jk} |k\rangle\langle k+i| a\rangle\langle b| \right) \\ &= \frac{1}{d} \omega^{-bj} \langle b+i|a\rangle, \\ (3.6) \quad &= \frac{1}{d} \omega^{-bj} \delta_{b+i,a}, \end{aligned}$$

where  $\delta_{i,j}$  is the Kronecker delta;

$$(3.7) \quad \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

We show that with  $\xi_{ij}$  defined as these values then  $|a\rangle\langle b|$  is in the span of  $\mathcal{E}$ . Now,

$$\begin{aligned} E_{i,j}^\dagger |a\rangle\langle b| &= E_{i,j}^\dagger \sum \xi_{k,l} E_{k,l} \\ (3.8) \quad &= \xi_{i,j} I + \sum_{k,l \neq i,j} \xi_{k,l} E_{i,j}^\dagger E_{k,l} \end{aligned}$$

where  $E_{i,j}^\dagger E_{k,l}$  has vanishing trace. Since

$$\begin{aligned}
 \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \frac{1}{d} \omega^{-bj} \delta_{b+i,a} E_{i,j} &= \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \frac{1}{d} \omega^{-bj} \delta_{b+i,a} \left( \sum_{k=0}^{d-1} \omega^{jk} |k+i\rangle \langle k| \right) \\
 &= \sum_{k=0}^{d-1} \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \frac{1}{d} \omega^{(k-b)j} \delta_{b+i,a} |k+i\rangle \langle k| \\
 &= \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} \frac{1}{d} \omega^{(k-b)j} |k+a-b\rangle \langle k| \\
 (3.9) \qquad \qquad \qquad &= |a\rangle \langle b|
 \end{aligned}$$

as  $\sum_j \frac{1}{d} \omega^{(k-b)j} = \delta_{k,b}$ , then  $\langle b | \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \xi_{i,j} E_{i,j} | a \rangle = \delta_{a,b}$ . Hence,  $|a\rangle \langle b| = \sum_{(i,j) \in \mathbb{Z}_d \times \mathbb{Z}_d} \xi_{i,j} E_{i,j}$  and the result follows.

#### 4. QUANTUM QUDIT TELEPORTATION

We now consider the transmission of quantum information with respect to a quantum noisy channel where a full continuum of noise is maintained. While classical information can be transmitted and protected from the effects of noise by replication, quantum information cannot be copied with perfect fidelity (Dieks (1982), Wootters and Zurek (1982)). Introduced by Bennett *et al.* (1983), *quantum teleportation* is an experimental demonstration of the means by which quantum communication is made possible and purports a fundamental distinction between quantum and classical information theory. Such distinction is maintained by the Bell-EPR correlations whereby an essential nonlocality principle, described by quantum entanglement, is revealed. This result was demonstrated experimentally by Aspect *et al.* (1982). Quantum teleportation takes advantage of the non-local behaviour of quantum mechanics by treating quantum entanglement as an information resource. While the Church-Turing Principle maintains that it is impossible to transmit quantum information by implementing a classical computation, Bennett *et al.* (1983) introduced quantum teleportation to overcome this limitation by developing a quantum algorithm that describes a complete communication transmission of quantum information. The first complete transmission of quantum information was performed by Nielsen *et al.* (1998). In the quantum teleportation protocol two parties  $\mathcal{A}$  and  $\mathcal{B}$  share a pair of particles in a maximally entangled state. If we suppose that  $\mathcal{A}$  is presented with a quantum system in an unknown quantum state  $|\psi\rangle$  then  $\mathcal{A}$  can make  $|\psi\rangle$  re-appear at  $\mathcal{B}$ 's location (see Fig. 2). Central to the protocol is the use of entanglement to transmit the quantum information of the unknown state between the parties. We now describe the protocol.

Quantum teleportation describes how two parties,  $\mathcal{A}$  and  $\mathcal{B}$ , process and communicate quantum information in a manner secure from the effects of error. Suppose  $\mathcal{A}$  wishes to communicate the state  $|\psi\rangle$  then the goal of teleportation is to transmit that particular quantum information state to  $\mathcal{B}$ . Further suppose that  $\mathcal{A}$  prepares the qudit  $|A\rangle$  where  $|A\rangle \in \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ . Similarly,  $\mathcal{B}$  prepares the qudit  $|B\rangle$  where  $|B\rangle \in \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ . In order to achieve teleportation,  $\mathcal{A}$  must interact the information state  $|\psi\rangle$  with a two qudit entangled state,  $|\beta_{AB}\rangle$ . The entangled state  $|\beta_{AB}\rangle$  is called a *generalised Bell state* whereby  $\mathcal{A}$  and  $\mathcal{B}$  each possess one qudit of this two qudit state. To construct a generalised Bell state  $|\beta_{AB}\rangle$ ,

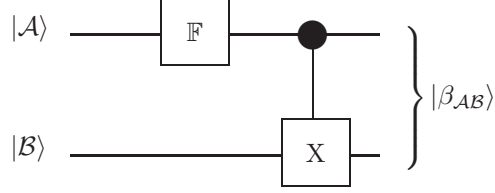


FIGURE 1. Generalised Bell State.

we first apply the Fourier transform  $\mathbb{F} \otimes I$  to the qudit  $|A\rangle$ . This acts on basis states  $|j\rangle |k\rangle$  as follows  $(\mathbb{F} \otimes I) |j\rangle |k\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{ij} |i\rangle |k\rangle$  where  $\omega$  is a primitive  $d^{\text{th}}$  root of unity in  $\mathbb{C}$  such that  $\omega^d = 1$  and  $\omega^t \neq 1$  for all  $0 < t < d$ . Secondly, we follow the Fourier transform by the controlled-NOT operation given by  $|k\rangle |l\rangle \mapsto |k\rangle |l+k \pmod{d}\rangle$  for all basis states  $|k\rangle |l\rangle$  which maps the two qudit state accordingly. Consequently, any pair of qudits  $|A\rangle |B\rangle$  from the  $d^2$  computational basis states of  $\mathbb{C}^d \otimes \mathbb{C}^d$  generate a generalised Bell state. In particular, applying the Fourier transform to the first half of the pair of qudit states  $|A\rangle |B\rangle$ , we obtain,

$$\begin{aligned}
 & \left( \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{x=0}^{d-1} \omega^{ix} |x\rangle |j\rangle \langle i| \langle j| \right) |A\rangle |B\rangle \\
 &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{x=0}^{d-1} \omega^{ix} |x\rangle |j\rangle \langle i| \langle A| \langle j| B\rangle \\
 (4.1) \quad &= \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} \omega^{Ax} |x\rangle |B\rangle.
 \end{aligned}$$

The action of the controlled-NOT operator on resulting state (4.1) completes the generalised Bell state construction

$$\begin{aligned}
 & \left( \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} |k\rangle |l+k\rangle \langle k| \langle l| \right) \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} \omega^{Ax} |x\rangle |B\rangle \\
 &= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \sum_{x=0}^{d-1} \omega^{Ax} |k\rangle |l+k\rangle \langle k|x\rangle \langle l| B\rangle \\
 &= \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} \omega^{Ax} |x\rangle |B+x\rangle \\
 (4.2) \quad &= |\beta_{AB}\rangle.
 \end{aligned}$$

Since the Bell pair is an entangled state Nielsen and Chuang (2000), any operator acting on the first qudit held by  $\mathcal{A}$  influences the state of the second qudit held by  $\mathcal{B}$ . This condition permits the teleportation of the quantum information state  $|\psi\rangle$

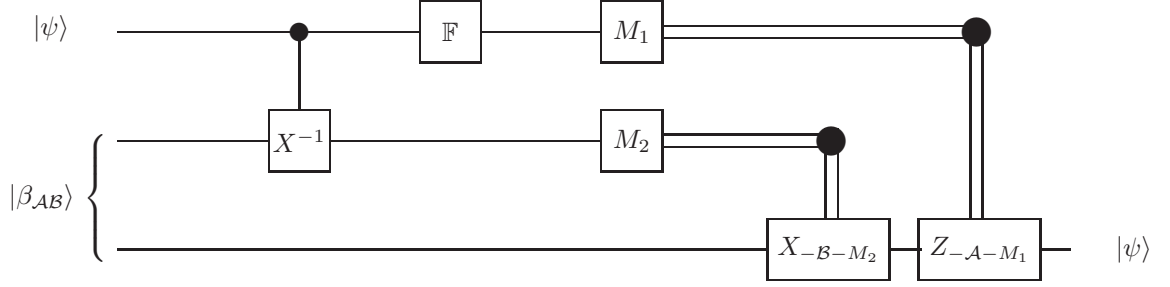


FIGURE 2. Quantum channel for teleporting a qudit.

between parties  $\mathcal{A}$  and  $\mathcal{B}$  when  $\mathcal{A}$  interacts  $|\psi\rangle$  with the first half of the generalised Bell pair (4.2). To negate the effects of the Bell state transformations on  $|\psi\rangle$ , thereby allowing the teleportation of  $|\psi\rangle$ ,  $\mathcal{A}$  transforms  $|\psi\rangle$  by applying the inverse of the generalised controlled-NOT operator for qudit states which is then followed by an application of the inverse Fourier transform. Now, the Fourier transform is unitary so its inverse is its adjoint, and the inverse of the generalised controlled-NOT operation has its action defined as  $|k\rangle|l\rangle \mapsto |k\rangle|l-k \pmod{d}\rangle$ . We write the state of the quantum system held by  $\mathcal{A}$  and  $\mathcal{B}$ , as

$$(4.3) \quad |\psi\rangle|\beta_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \alpha_a |a\rangle \left( \sum_{x=0}^{d-1} \omega^{\mathcal{A}x} |x\rangle |\mathcal{B}+x\rangle \right).$$

$\mathcal{A}$  initiates teleportation of the quantum information state  $|\psi\rangle$  by applying the inverse generalised controlled-NOT operation between  $|\psi\rangle$  and the qudit of the generalised Bell state held by  $\mathcal{A}$ , thereby obtaining,

$$(4.4) \quad \begin{aligned} & \left( \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \sum_{m=0}^{d-1} |k\rangle|l-k\rangle|m\rangle\langle k|\langle l|\langle m| \right) \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \alpha_a |a\rangle \sum_{x=0}^{d-1} \omega^{\mathcal{A}x} |x\rangle |\mathcal{B}+x\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \sum_{m=0}^{d-1} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \alpha_a \omega^{\mathcal{A}x} |k\rangle|l-k\rangle|m\rangle\langle k|a\rangle\langle l|x\rangle\langle m|\mathcal{B}+x\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \alpha_a \omega^{\mathcal{A}x} |a\rangle|x-a\rangle|\mathcal{B}+x\rangle. \end{aligned}$$

Following this result,  $\mathcal{A}$  applies the discrete Fourier transform on the first qudit of the state (4.4). The outcome of this operation is to place the state (4.4) into the state given by

$$\left( \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{y=0}^{d-1} \sum_{j=0}^{d-1} \sum_{n=0}^{d-1} \omega^{iy} |y\rangle|j\rangle|n\rangle\langle i|\langle j|\langle n| \right) \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \alpha_a \omega^{\mathcal{A}x} |a\rangle|x-a\rangle|\mathcal{B}+x\rangle$$

$$\begin{aligned}
&= \frac{1}{d} \sum_{i=0}^{d-1} \sum_{y=0}^{d-1} \sum_{j=0}^{d-1} \sum_{n=0}^{d-1} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \alpha_a \omega^{iy} \omega^{\mathcal{A}x} |y\rangle |j\rangle |n\rangle \langle i|a\rangle \langle j|x-a\rangle \langle n|\mathcal{B}+x\rangle \\
&= \frac{1}{d} \sum_{y=0}^{d-1} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \alpha_a \omega^{ay} \omega^{\mathcal{A}x} |y\rangle |x-a\rangle |\mathcal{B}+x\rangle \\
&= \frac{1}{d} \sum_{y=0}^{d-1} \sum_{a=0}^{d-1} \sum_{x=0}^{d-1} \sum_{z=0}^{d-1} \alpha_a \omega^{ay} \omega^{\mathcal{A}x} |y\rangle |z\rangle \langle z|x-a\rangle |\mathcal{B}+x\rangle \\
&= \frac{1}{d} \sum_{y=0}^{d-1} \sum_{a=0}^{d-1} \sum_{z=0}^{d-1} \alpha_a \omega^{ay} \omega^{\mathcal{A}(z+a)} |y\rangle |z\rangle |\mathcal{B}+z+a\rangle \\
(4.5) \quad &= \frac{1}{d} \sum_{y=0}^{d-1} \sum_{z=0}^{d-1} \omega^{\mathcal{A}z} |y\rangle |z\rangle \left( \sum_{a=0}^{d-1} \alpha_a \omega^{a(y+\mathcal{A})} |\mathcal{B}+z+a\rangle \right).
\end{aligned}$$

The qudit of the generalised Bell state held by  $\mathcal{B}$  is transformed into the state  $\sum_{a=0}^{d-1} \alpha_a \omega^{a(y+\mathcal{A})} |\mathcal{B}+z+a\rangle$ . Thus  $\mathcal{A}$  has teleported a quantum information state  $|\psi'\rangle$  to  $\mathcal{B}$ , however, it has been subjected to *error* over the channel and therefore  $\mathcal{B}$  receives  $\sum_{a=0}^{d-1} \alpha_a \omega^{a(y+\mathcal{A})} |\mathcal{B}+z+a\rangle$  instead of  $\sum_{a=0}^{d-1} \alpha_a |a\rangle$ . A measurement projection onto the computational basis state of  $\mathbb{C}^d \otimes \mathbb{C}^d$  is performed by  $\mathcal{A}$  on the first and second qudit of the state of the quantum system (4.5) which yields two classical numbers. Simultaneously, the third qudit of the state of the system (4.5) teleported to  $\mathcal{B}$  collapses to a post-measurement state that is dependent upon the measurement outcome obtained by  $\mathcal{A}$ . Let  $M_1, M_2$  be two classical numbers corresponding to the resulting states  $|M_1\rangle |M_2\rangle$ . Then the state of the qudit held by  $\mathcal{B}$  is given by  $\sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |\mathcal{B}+M_2+a\rangle$ . The set  $M_1, M_2$  is transferred by classical means to  $\mathcal{B}$ , where upon delivery  $\mathcal{B}$  learns which of the generalised Pauli operators are required to correct the effect of the error. In particular,  $\mathcal{B}$  applies the operators

$$X_{-\mathcal{B}-M_2} = \sum_{x=0}^{d-1} |x-\mathcal{B}-M_2\rangle \langle x|$$

and

$$Z_{-\mathcal{A}-M_1} = \sum_{z=0}^{d-1} \omega^{(-\mathcal{A}-M_1)z} |z\rangle \langle z|$$

in order to return the post-measurement state  $\sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |\mathcal{B}+M_2+a\rangle$  to the initial quantum information state  $|\psi\rangle$ . Hence, applying  $X_{-\mathcal{B}-M_2}$  to the post-measurement state,  $\mathcal{B}$  obtains

$$\begin{aligned}
&X_{-\mathcal{B}-M_2} \left( \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |\mathcal{B}+M_2+a\rangle \right) \\
&= \sum_{x=0}^{d-1} |x-\mathcal{B}-M_2\rangle \langle x| \left( \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |\mathcal{B}+M_2+a\rangle \right)
\end{aligned}$$



$$\begin{aligned}
&= \sum_{x=0}^{d-1} \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |x - \mathcal{B} - M_2\rangle \langle x | \mathcal{B} + M_2 + a\rangle \\
(4.6) \quad &= \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |a\rangle.
\end{aligned}$$

The operator  $Z_{-\mathcal{A}-M_1}$  is then applied by  $\mathcal{B}$  on result (4.6) returning the post-measurement state to the quantum information state initially held by  $\mathcal{A}$ ,

$$\begin{aligned}
&Z_{-\mathcal{A}-M_1} \left( \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} |a\rangle \right) \\
&= \sum_{z=0}^{d-1} \omega^{(-\mathcal{A}-M_1)z} |z\rangle \langle z| \left( \sum_{a=0}^{d-1} \alpha_a \omega^{(M_1+\mathcal{A})a} |a\rangle \right) \\
&= \sum_{z=0}^{d-1} \sum_{a=0}^{d-1} \alpha_a \omega^{(\mathcal{A}+M_1)a} \omega^{(-\mathcal{A}-M_1)z} |z\rangle \langle z| a\rangle \\
(4.7) \quad &= \sum_{a=0}^{d-1} \alpha_a |a\rangle.
\end{aligned}$$

This ends the teleportation protocol - the state of  $\mathcal{B}$ 's system is left in the same state as the one initially presented to  $\mathcal{A}$ . The quantum information can only be obtained if it vanishes from  $\mathcal{A}$  thereby upholding the no-cloning theorem (Dieks (1982), Wootters and Zurek (1982)). Thus  $\mathcal{B}$  obtains the quantum information which  $\mathcal{A}$  wished to transmit. This is what it means for the quantum information to have been transmitted (Timpson (2006)).

## REFERENCES

- [1] Aspect A, Dalibard , and Roger G (1982), *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers*, Physical Review Letters, Vol. 49, pp. 1804-1807.
- [2] Bell J (1964), *On the Einstein-Podolsky-Rosen Paradox*, Physics, Vol. 1, pp. 195-200.
- [3] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters A K (1983), *Teleporting an Unknown Quantum State via Dual Classical and EPR Channels*, Physical Review Letters, Vol. 70, pp. 1895-1899.
- [4] Deutsch D (1985), *Quantum theory, the Church-Turing principle and the universal quantum computer*, Proc. Roy. Soc. Lond. A, Vol. 400, pp.97-117.
- [5] Deutsch D (1989), *Quantum Computational Networks* Proc. Roy. Soc. Lond. A, Vol. 425, pp. 73-90.
- [6] Dieks D (1982), *Communication by EPR devices* Phys. Lett. A, Vol. 92(6), pp. 271-272.
- [7] Nielsen M A and Chuang I L (2000), *Quantum Computations and Quantum Information*, Cambridge University Press, 2000.
- [8] Nielsen A M, Knill E and Laflamme R (1998), *Complete Quantum Teleportation using Nuclear Magnetic Resonance*, Nature, Vol. 396, No. 7706, pp. 52-55.
- [9] Preskill J (1998), *Quantum Computing: Pro and Con*, Proc. Roy. Soc. Lond. A Vol. 454, pp. 469-486. LANL e-print, quant-ph/9705032.
- [10] Shannon C E (1948), *A Mathematical Theory of Communication*, Bell System Technical Journal, Vol. 27, pp. 379-423.
- [11] Shor P W (1995), *Scheme for reducing decoherence in quantum computer memory*, Physical Review A, Vol. 52, pp. 2493-2496.
- [12] Timpson C G (2006), *Philosophical Aspects of Quantum Information Theory*, quant-ph/0611187.
- [13] Werner R (2001), *All teleportation and dense coding schemes*, J. Phys. A, vol. 34, pp. 7081-7094, quant-ph/0003070.

- [14] Weyl H (1931), *The Thoery of Groups and Quantum Mechanics*, Dover Publications, New York.
- [15] Wootters W K and Zurek W Z (1982), *A Single Quantum Cannot be Cloned*, Nature 299, pp. 802-803.
- [16] Zanardi P, Zalka C and Faoro L (2000), *Entangling power of quantum evolutions*, Phys. Review A, 62, 030301.

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND

DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY, UNIVERSITY OF LONDON, EGHAM, SURREY, TW20 0EX, UK